

THEORY OF COUPLED LANDAU DAMPING

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The influence of linear coupling between the transverse planes on Landau damping of coherent instabilities is assessed using two typical frequency distributions (Lorentzian, $\rho(f) \propto 1/(1+u^2)$, and “elliptical”, $\rho(f) \propto \sqrt{1-u^2}$ where $u = (f-f_0)/\Delta f$). A general stability criterion is derived in both cases that includes the coupling strength and the distance from the resonance $Q_h - Q_v = \text{integer}$. It reveals the possibility of sharing the “stabilising” frequency spreads between the two planes. This can significantly improve the coherent beam stability, especially in cases where the situation is more critical in one plane. Another important observation is the fact that the influence of a large imaginary part in the beam-environment impedance, which normally requires a large frequency spread for Landau damping, can be compensated (at least in one plane) by a judicious choice of the coupling. The conjunction of these two features could explain why a machine like the CERN-PS can be stabilised by tuning close to a coupling resonance and can be used to determine optimum values for the tune split and the coupling strength.

Keywords: Landau damping; Coherent instabilities; Linear coupling

1 INTRODUCTION

The energy exchange between the degrees of freedom of a multi-dimensional oscillator is a widespread phenomenon in physics. Strong coupling between the transverse planes of a particle beam leads to an “equi-partition” of the oscillation energy, including the instability growth rates in the case of coherent instability. The purpose of this report is to show that, in the presence of a frequency spread, in addition to the exchange of energy, there can also be a partition of Landau damping for “optimum” coupling.

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Section 2 is devoted to coasting beams. The results are then extended to bunched beams in Section 3. The sharing of both Landau damping and chromaticity is illustrated in the case of head–tail instabilities using the “hollow-bunch model” and the transfer of damping by feedbacks is also discussed.

2 COASTING BEAMS

2.1 Equations of Motion

Four types of forces are taken into account in writing the equations of betatron motion of a test particle (i):¹

- (1) the external focusing forces that depend on the horizontal deviation x_i (the vertical deviation y_i) of the particle from a fixed reference (e.g. the centre of the chamber). The corresponding tune is $Q_{0x,i}$ ($Q_{0y,i}$);
- (2) the coherent space-charge forces that depend on the deviation \bar{x} (\bar{y}) of the beam centre from the centre of the chamber. The corresponding tune shift is $\Delta Q_{\text{coh},x}$ ($\Delta Q_{\text{coh},y}$);
- (3) the incoherent space-charge forces that depend on the deviation $x_i - \bar{x}$ ($y_i - \bar{y}$) of the particle from the beam centre. The corresponding tune shift is $\Delta Q_{\text{inc},x}$ ($\Delta Q_{\text{inc},y}$);
- (4) the coupling forces, represented by the normalised skew gradient $\underline{K}_i = (e/p_i)(\partial B_{x,i}/\partial x_i)$, that make the vertical (horizontal) deviation appear in the horizontal (vertical) equation of motion of the particle. This yields

$$\begin{aligned}
 \frac{d^2 x_i}{dt^2} + \Omega_i^2 (Q_{0x,i}^2 + 2Q_{0x,i} \Delta Q_{\text{inc},x}) x_i \\
 = -2\Omega_i^2 Q_{0x,i} (\Delta Q_{\text{coh},x} - \Delta Q_{\text{inc},x}) \bar{x} + \underline{K}_i R^2 \Omega_i^2 y_i, \\
 \frac{d^2 y_i}{dt^2} + \Omega_i^2 (Q_{0y,i}^2 + 2Q_{0y,i} \Delta Q_{\text{inc},y}) y_i \\
 = -2\Omega_i^2 Q_{0y,i} (\Delta Q_{\text{coh},y} - \Delta Q_{\text{inc},y}) \bar{y} + \underline{K}_i R^2 \Omega_i^2 x_i.
 \end{aligned} \tag{1}$$

Here, e is the elementary charge, p_i the momentum of the particle, $B_{x,i}$ the horizontal magnetic field at the position of the particle, t the time, Ω_i the revolution frequency of the particle and R the average radius of the machine. The coherent and incoherent Laslett tune shifts are generalised

here to include wake fields. The first three forces can be treated in the smooth approximation for simplicity. Using the normalised (Courant–Snyder) coordinates and angle given by $\eta_i = x_i/\beta_{0x,i}^{1/2}(s)$, $\zeta_i = y_i/\beta_{0y,i}^{1/2}(s)$ and $\phi_i = Q_{0x,i}^{-1} \int \beta_{0x,i}^{-1}(s) ds \approx Q_{0y,i}^{-1} \int \beta_{0y,i}^{-1}(s) ds$, where the betatron functions are given by $\beta_{0x,i} \approx R/Q_{0x,i} \approx R/Q_{0x0}$ and $\beta_{0y,i} \approx R/Q_{0y,i} \approx R/Q_{0y0}$ in the smooth approximation (with Q_{0x0} and Q_{0y0} the averages of the incoherent tunes), Eqs. (1) can be written

$$\begin{aligned} \frac{d^2 \eta_i}{d\phi^2} + Q_{x,i}^2 \eta_i &= -j \frac{e\beta I Z_x}{2\pi R m_0 \gamma \Omega_0^2} \bar{\eta} + R^2 \left(\frac{Q_{0x0}}{Q_{0y0}} \right)^{1/2} \underline{K}_0 \zeta_i, \\ \frac{d^2 \zeta_i}{d\phi^2} + Q_{y,i}^2 \zeta_i &= -j \frac{e\beta I Z_y}{2\pi R m_0 \gamma \Omega_0^2} \bar{\zeta} + R^2 \left(\frac{Q_{0y0}}{Q_{0x0}} \right)^{1/2} \underline{K}_0 \eta_i. \end{aligned} \quad (2)$$

Here, s is the azimuthal coordinate, $Q_{x,i} = Q_{0x,i} + \Delta Q_{\text{inc},x}$ and $Q_{y,i} = Q_{0y,i} + \Delta Q_{\text{inc},y}$ are the new incoherent tunes, $j = \sqrt{-1}$ is the imaginary unit, β and γ are the relativistic velocity and mass factors, I is the circulating beam current, m_0 is the proton rest mass, $\bar{\eta}$ and $\bar{\zeta}$ are the normalised coordinates of the beam centre. Furthermore, the revolution frequency Ω_i has been replaced by its average value Ω_0 ($\phi_i = \Omega_0 t = \phi$) and the skew gradient has been supposed to be the same for all the particles, i.e. $\underline{K}_i = \underline{K}_0$. Finally, the coherent and incoherent tune shifts have been replaced by the transverse coupling impedances which are given by, e.g. in the horizontal plane,²

$$Z_x(\omega) = \frac{j}{e\beta I \Delta_x} \int_0^{2\pi R} \bar{F}_x ds \approx -j \frac{2\pi R m_0 \gamma}{e\beta I} 2\Omega_0^2 Q_{0x0} (\Delta Q_{\text{coh},x} - \Delta Q_{\text{inc},x})_{(\omega)}, \quad (3)$$

where $\Delta_x e^{j\omega t}$ describes the horizontal beam oscillations and $\bar{F}_x e^{j\omega t}$ is the average of the horizontal force over the beam cross section.

2.2 Dispersion Relation

In the following, transverse betatron frequency spreads specified by externally-given beam frequency spectra are assumed. The ensemble of particles has spectra with the distribution functions $\rho_x(\omega_{x,i})$ and $\rho_y(\omega_{y,i})$ which are supposed to be uncorrelated and normalised to unity. The betatron frequencies are given by $\omega_{x,i} = \Omega_0 Q_{x,i}$ and $\omega_{y,i} = \Omega_0 Q_{y,i}$. Moreover, in a circular machine, linear coupling is periodic in ϕ with

period 2π , and thus can be expanded into Fourier series

$$\underline{K}_0(\phi) = \sum_{l=-\infty}^{l=+\infty} \hat{\underline{K}}_0(l) e^{jl\phi}, \quad (4)$$

with

$$\hat{\underline{K}}_0(l) = \frac{1}{2\pi} \int_0^{2\pi} \underline{K}_0(\phi) e^{-jl\phi} d\phi. \quad (5)$$

Considering only the dominant Fourier component of the coupling (l) and following the standard procedure of identifying normal mode frequencies, yields particular solutions of the form

$$\eta_i = H_i e^{jQ_c \phi}, \quad \zeta_i = Z_i e^{j(Q_c - l)\phi}, \quad (6)$$

where Q_c is the coherent tune to be determined. Substituting Eqs. (6) into Eqs. (2) and integrating over the transverse spectra, yields the two-dimensional dispersion relation in the coherent betatron frequency $\omega_c = \Omega_0 Q_c$

$$\left[\left(\int_{-\infty}^{+\infty} \frac{\rho_x(\omega_{x,i}) d\omega_{x,i}}{\omega_c - \omega_{x,i}} \right)^{-1} - U_x + jV_x \right] \times \left[\left(\int_{-\infty}^{+\infty} \frac{\rho_y(\omega_{y,i}) d\omega_{y,i}}{\omega_c - \Omega_0 - \omega_{y,i}} \right)^{-1} - U_y + jV_y \right] = \frac{|\hat{\underline{K}}_0(l)|^2 R^4 \Omega_0^4}{4\omega_{x0}\omega_{y0}}, \quad (7)$$

making the usual assumptions $\omega_{x,i} \approx \omega_c \approx \omega_{x0}$ and $\omega_{y,i} \approx \omega_c - \Omega_0 \approx \omega_{y0}$, where $\omega_{x0} = \Omega_0 Q_{x0}$ and $\omega_{y0} = \Omega_0 Q_{y0}$ are the centres of the distributions. Here, the dispersion relation coefficients $U_{x,y}$ ^a and $V_{x,y}$ of Laslett *et al.*³ have been used. They are related to the coupling impedances by⁴

$$(U_{x,y} - jV_{x,y})_{(\omega)} \approx \frac{j e \beta I Z_{x,y}(\omega)}{4\pi R m_0 \gamma \omega_{x0,y0}}. \quad (8)$$

One has to remember that the wake field terms must be evaluated at the local collective frequencies (frequencies seen at fixed locations around the accelerator), given by

$$\omega_1 \approx (n_x + Q_{x0})\Omega_0, \quad \omega_2 \approx (n_y + Q_{y0})\Omega_0, \quad (9)$$

where $n_{x,y}$ are the azimuthal mode numbers related by $n_x = n_y - l$.

^a Sometimes $U + V$ is written in place of U as used here.

It is known that the treatment of Landau damping by non-linearities is more involved and that, in the plane of coherent motion, the derivative of the distribution function with respect to the oscillation amplitude appears in the dispersion integral instead of the distribution function itself. A fairly straightforward way to get the right answer is to use the Vlasov equation.³ Using the single particle equation formalism, Hereward has obtained the same result⁵ considering "second order" non-linear terms. This result (Eq. (5.4) of Ref. 5) can also be generalised in the presence of coupling, and the two-dimensional dispersion relation is similar to Eq. (7). Indeed, introducing the functions $h_x(\hat{x}_i)$ and $h_y(\hat{y}_i)$ to describe the distributions of the incoherent betatron amplitudes and choosing the normalisation

$$\int_0^{+\infty} h_x(\hat{x}_i) \hat{x}_i d\hat{x}_i = 1, \quad \int_0^{+\infty} h_y(\hat{y}_i) \hat{y}_i d\hat{y}_i = 1, \quad (10)$$

the two-dimensional dispersion relation reads

$$\begin{aligned} & \left\{ \left[\int_0^{+\infty} \frac{1}{\omega_c - \omega_{x,i}(\hat{x}_i)} \left(-\frac{h'_x(\hat{x}_i) \hat{x}_i^2}{2} \right) d\hat{x}_i \right]^{-1} - U_x + jV_x \right\} \\ & \times \left\{ \left[\int_0^{+\infty} \frac{1}{\omega_c - \Omega_0 - \omega_{y,i}(\hat{y}_i)} \left(-\frac{h'_y(\hat{y}_i) \hat{y}_i^2}{2} \right) d\hat{y}_i \right]^{-1} - U_y + jV_y \right\} \\ & = \frac{|\underline{\hat{K}}_0(l)|^2 R^4 \Omega_0^4}{4\omega_{x0}\omega_{y0}}. \end{aligned} \quad (11)$$

In the following, Eq. (7) is discussed but the same analysis applies for Eq. (11).

2.3 Stability Criterion for Lorentzian Spectra

In this case, the distribution function, e.g. in the horizontal plane, from Ref. 6 is

$$\rho_x(\omega_{x,i}) = \frac{\delta\omega_x}{\pi} [(\omega_{x,i} - \omega_{x0})^2 + \delta\omega_x^2]^{-1}, \quad (12)$$

where $\delta\omega_x$ is the half width at half maximum of the spectrum. The corresponding dispersion integral is given by⁶

$$\int_{-\infty}^{+\infty} \frac{\rho_x(\omega_{x,i}) d\omega_{x,i}}{\omega_c - \omega_{x,i}} = \frac{1}{\omega_c - \omega_{x0} - j\delta\omega_x}. \quad (13)$$

Substituting Eq. (13), and the similar equation for the vertical plane, into Eq. (7), the dispersion relation becomes

$$[\omega_c - \omega_{x0} - U_x - j(\delta\omega_x - V_x)] \times [\omega_c - \omega_{y0} - \Omega_0 - U_y - j(\delta\omega_y - V_y)] = \frac{|\hat{K}_0(l)|^2 R^4 \Omega_0^4}{4\omega_{x0}\omega_{y0}}. \quad (14)$$

The dispersion equation has two solutions for ω_c which describe the two coherent oscillation modes of the coupled system. Coherent motions of the form $e^{j\omega_c t}$ are considered; therefore, for each solution ω_c , $\text{Re}(\omega_c)$ describes the coherent oscillation frequency and $-\text{Im}(\omega_c)$ describes the instability growth rate. Thus, to be stable, a coherent oscillation mode must satisfy $\text{Im}(\omega_c) \geq 0$.

The imaginary parts of the two coherent oscillation frequencies are given by

$$\text{Im}(\omega_{c1,2}) = (\delta\omega_{x,y} - V_{x,y}) \pm \frac{(\delta\omega_y - V_y - \delta\omega_x + V_x)}{2} C(a, \delta). \quad (15)$$

Here, $C(a, \delta)$ is a normalised coupling (or sharing) function given by

$$C(a, \delta) = 1 - \frac{1}{\sqrt{2}} \sqrt{1 - 4a^2 - \delta^2 + \sqrt{(-1 + 4a^2 + \delta^2)^2 + 4\delta^2}}, \quad (16)$$

with

$$a = \frac{|\hat{K}_0(l)| R^2 \Omega_0^2}{2\sqrt{\omega_{x0}\omega_{y0}}|\delta\omega_y - V_y - \delta\omega_x + V_x|}, \quad \delta = \frac{\Omega_0|Q_h - Q_v - l|}{|\delta\omega_y - V_y - \delta\omega_x + V_x|}, \quad (17)$$

where

$$Q_{h,v} = (\omega_{x0,y0} + U_{x,y})/\Omega_0 \quad (18)$$

are the horizontal and vertical coherent tunes in the presence of wake fields ($U_{x,y}$), but in the absence of coupling.

Three plots of $C(a)$ are represented in Figure 1 for $\delta = 1$, $\delta = 0.25$ and $\delta = 0$. Whatever the value of δ , the sharing function varies between $C = 0$

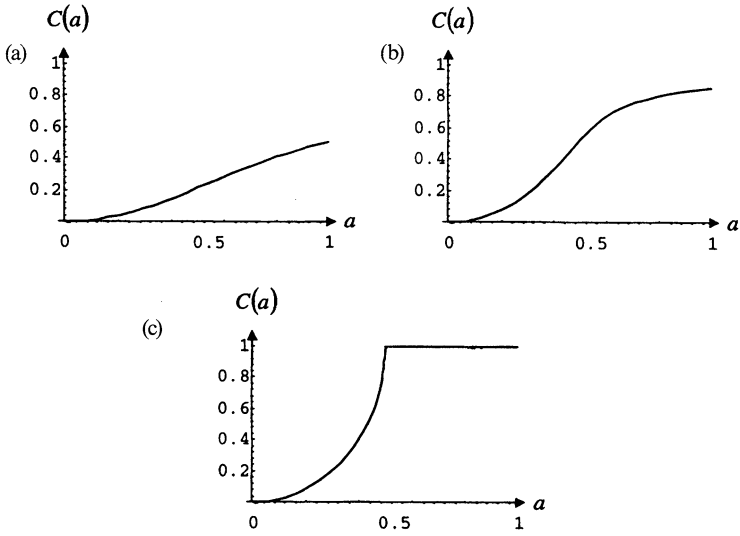


FIGURE 1 Sharing function $C(a)$ for: (a) $\delta = 1$; (b) $\delta = 0.25$; (c) $\delta = 0$.

and $C = 1$. However, the rate at which $C(a)$ grows increases when δ decreases. The smaller the tune separation, the easier the sharing. The sharing ratio can be chosen by adjusting the tune split and/or the coupling strength.

For $C = 0$ (no coupling), transverse stability in the horizontal and vertical planes requires

$$\delta\omega_x \geq V_x, \quad \delta\omega_y \geq V_y. \quad (19)$$

For $C = 1$ (full coupling), by virtue of Eq. (15), these two criteria reduce into the stability criterion

$$\delta\omega_x + \delta\omega_y \geq V_x + V_y. \quad (20)$$

Equation (20) shows the beneficial effect of coupling. Even in the absence of a frequency spread in one plane, a coherent instability can be damped thanks to the other plane: Landau damping is transferred from the stable to the unstable plane. In the case of full coupling, each plane has the mean transverse betatron frequency spread $(\delta\omega_x + \delta\omega_y)/2$ to damp the instability represented by the mean instability growth rate $(V_x + V_y)/2$.

When the transverse tune spreads are equal, there is no re-distribution of Landau damping but there still is a sharing of the growth rates. Therefore, the beam can be stabilised provided that $\delta\omega_x = \delta\omega_y \geq (V_x + V_y)/2$. The result of coupling is thus a transfer of Landau damping from the stable to the unstable plane and at the same time a transfer of the instability growth rate from the unstable to the stable plane up to a perfect sharing of both damping and growth. It can be seen from Eq. (20), that if the two planes are stable without coupling, then they remain stable with full coupling. In the same way, if both planes are unstable without coupling, they remain unstable with full coupling.

Consider the interesting case of one unstable transverse plane in the absence of coupling. If the necessary condition of Eq. (20) is fulfilled, then it is possible to stabilise the beam in the two planes by choosing a pair (a, δ) that satisfies $\text{Im}(\omega_{c1,2}) \geq 0$ (see Eq. (15)). The stabilising values of the modulus of the Fourier coefficient of the skew gradient are given by

$$|\hat{\underline{K}}_0(l)| \geq \frac{2[-Q_{x0}Q_{y0}(\delta\omega_x - V_x)(\delta\omega_y - V_y)]^{1/2}}{R^2\Omega_0} \times \frac{[(\delta\omega_x + \delta\omega_y - V_x - V_y)^2 + \Omega_0^2(Q_h - Q_v - l)^2]^{1/2}}{\delta\omega_x + \delta\omega_y - V_x - V_y}. \quad (21)$$

The plot of the stability boundary (given by the equal sign in Eq. (21)) is a symmetric curve with respect to the vertical axis (Figure 2). It exhibits one stability region for the modulus of the Fourier component of the skew gradient and the tune split $Q_h - Q_v - l$.

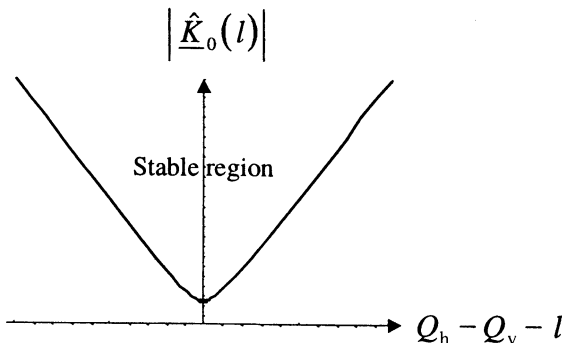


FIGURE 2 Shape of stability boundary in the plane $|\hat{\underline{K}}_0(l)|$ vs. $Q_h - Q_v - l$.

2.4 Stability Criterion for Elliptical Spectra

Due to its infinite tails, the Lorentzian frequency distribution tends to underestimate two important points. The first is the effect of the real betatron frequency shift. The importance of U emerges already in the uncoupled case where, for distributions without long tails, Landau damping is prevented when the shift $|U|$ is larger than the frequency spread $\Delta\omega$. This is explained by the large detuning which shifts the coherent frequency $\omega_0 + U$ to a value outside the spectrum $\omega_0 \pm \Delta\omega$.

As a second point, which is in fact closely related to the first, it will be found that too strong coupling can be detrimental and may shift the coherent frequency outside the spectrum and thus again prevent Landau damping. To study these two effects, consider elliptical spectra, knowing that Lorentzian and elliptical spectra are limiting cases and that realistic distributions are probably between them.

In this case, the distribution function, e.g. in the horizontal plane, from Ref. 7 is

$$\rho_x(\omega_{x,i}) = \begin{cases} \frac{2}{\pi\Delta\omega_x^2} \sqrt{\Delta\omega_x^2 - (\omega_{x,i} - \omega_{x0})^2}, & |\omega_{x,i} - \omega_{x0}| \leq \Delta\omega_x, \\ 0, & |\omega_{x,i} - \omega_{x0}| > \Delta\omega_x. \end{cases} \quad (22)$$

Here, $\Delta\omega_x$ is the half width at the bottom of the distribution. The corresponding dispersion integral is given by⁷

$$\int_{-\infty}^{+\infty} \frac{\rho_x(\omega_{x,i}) d\omega_{x,i}}{\omega_c - \omega_{x,i}} = 2 \left[\omega_c - \omega_{x0} - j\sqrt{\Delta\omega_x^2 - (\omega_c - \omega_{x0})^2} \right]^{-1}, \quad (23)$$

with

$$\begin{aligned} & -j\sqrt{\Delta\omega_x^2 - (\omega_c - \omega_{x0})^2} \\ & = \begin{cases} \sqrt{(\omega_c - \omega_{x0})^2 - \Delta\omega_x^2} & \text{for } \omega_c > \omega_{x0} + \Delta\omega_x, \\ -\sqrt{(\omega_c - \omega_{x0})^2 - \Delta\omega_x^2} & \text{for } \omega_c < \omega_{x0} - \Delta\omega_x. \end{cases} \end{aligned} \quad (24)$$

Substituting Eq. (23), and the similar equation for the vertical plane, into Eq. (7), the dispersion relation becomes

$$\begin{aligned} & \left\{ \omega_c - \omega_{x0} - 2U_x - j \left[\sqrt{\Delta\omega_x^2 - (\omega_c - \omega_{x0})^2} - 2V_x \right] \right\} \\ & \times \left\{ \omega_c - \omega_{y0} - \Omega_0 - 2U_y - j \left[\sqrt{\Delta\omega_y^2 - (\omega_c - \omega_{y0} - \Omega_0)^2} - 2V_y \right] \right\} \\ & = \frac{|\hat{K}_0(l)|^2 R^4 \Omega_0^4}{\omega_{x0} \omega_{y0}}. \end{aligned} \quad (25)$$

It seems to be difficult to solve this equation in the general case but an approximate stability criterion, which can be checked numerically and is always close to the solution to within a few percent, can be expressed as follows. Two cases appear depending on whether the transverse coherent tunes (in the absence of coupling) are “far” from or “near” each other. These two terms will be explained at the end of this section.

(1) Q_h “far” from $Q_v + l$: In this case, the result of coupling is a sharing of the instability growth rates only. There is no transfer of Landau damping since the coherent tunes are too far from each other to share their stabilising spreads. The equation that has to be solved to obtain the stability criterion can be approximated by Eq. (25) with $\Delta\omega_{x,y} = 0$. The equation that is obtained is the same as Eq. (14) with $\delta\omega_{x,y} = 0$. By analogy with the Lorentzian case, the necessary condition for stability is (see Eq. (20))

$$V_x + V_y \leq 0. \quad (26)$$

If Eq. (26) is true then it is possible to stabilise the beam and the stability criterion is given by Eq. (21) with $\delta\omega_{x,y} = 0$.

(2) Q_h “near” $Q_v + l$: In this case, in addition to the sharing of the instability growth rates, there is also a transfer of Landau damping. Equation (25) is then approximated by⁸

$$\begin{aligned} & \left\{ \omega_c - \omega_{x0} - \frac{\Delta\omega_x^2 + 4U_x^2}{4U_x} - j \left[\text{Re} \left(\sqrt{\Delta\omega_x^2 - 4U_x^2} \right) - 2V_x \right] \right\} \\ & \times \left\{ \omega_c - \omega_{y0} - \Omega_0 - \frac{\Delta\omega_y^2 + 4U_y^2}{4U_y} - j \left[\text{Re} \left(\sqrt{\Delta\omega_y^2 - 4U_y^2} \right) - 2V_y \right] \right\} \\ & = \frac{|\hat{K}_0(l)|^2 R^4 \Omega_0^4}{\omega_{x0} \omega_{y0}}, \end{aligned} \quad (27)$$

where $\text{Re}()$ stands for real part, indicating that the square root has to be omitted if the argument under the root is negative. The necessary condition for stability is thus

$$\text{Re}\left(\sqrt{\Delta\omega_x^2 - 4U_x^2} + \sqrt{\Delta\omega_y^2 - 4U_y^2}\right) \geq 2(V_x + V_y). \quad (28)$$

If Eq. (28) is true then it is possible to stabilise the beam and a condition similar to Eq. (21) for the stabilising values of the coupling coefficient may be approximated by

$$|\hat{K}_0(l)| \approx \frac{\left\{-Q_{x0}Q_{y0}\left[\text{Re}\left(\sqrt{\Delta\omega_x^2 - 4U_x^2}\right) - 2V_x\right]\left[\text{Re}\left(\sqrt{\Delta\omega_y^2 - 4U_y^2}\right) - 2V_y\right]\right\}^{1/2}}{R^2\Omega_0}. \quad (29)$$

Equation (28) generalises the one-dimensional stability conditions which are written as

$$\text{Re}\left(\sqrt{\Delta\omega_x^2 - 4U_x^2}\right) \geq 2V_x, \quad \text{Re}\left(\sqrt{\Delta\omega_y^2 - 4U_y^2}\right) \geq 2V_y, \quad (30)$$

or equivalently (in the more familiar form, for $V_{x,y} > 0$)

$$\Delta\omega_x \geq 2|U_x - jV_x|, \quad \Delta\omega_y \geq 2|U_y - jV_y|. \quad (31)$$

The result (Eq. (28)) reveals the features mentioned earlier that the imaginary part (U) of the coupling impedance in the unstable plane can be “cancelled” by coupling. The frequency spreads are shared between the two planes and a large tune dispersion becomes effective also in the plane which, without coupling, has little spread.

Coming back to the terms “far” from and “near” to the coupling resonance used above and guided by the results of numerical solutions of Eq. (25), it can be concluded that the stability is obtained for coupling values in a range near the value given by Eq. (29) and that the tune separation $|Q_h - Q_v - l|$ should be smaller than the order of magnitude of $(\Delta\omega_x + \Delta\omega_y)/\Omega_0$ (which seems to be very small) in order to have the transfer of Landau damping.

3 EXTENSION TO BUNCHED BEAMS

3.1 Equations of Motion

In the case of rigid bunched beams,⁹ the equation of betatron motion of the i th particle in the b th bunch, e.g. in the horizontal plane and in the presence of both wake fields and linear coupling between the transverse planes, is given by¹

$$m_0\gamma\left(\frac{d^2x_i}{dt^2} + \omega_{x,i}^2x_i\right) = U_{1x}\left(\sum_b \frac{dP_{x,r}}{d\vartheta}\right)_b + \frac{W_x}{2\pi\sqrt{\Omega_0}} \sum_r x_r G(\vartheta_r - \vartheta_i, Q_{x,r}) \\ + m_0\gamma \underline{K}_i R^2 \Omega_i^2 y_i. \quad (32)$$

Here, U_{1x} and W_x are real coefficients proportional to the dispersion relation coefficients U_x and V_x respectively, and $(\sum_b dP_{x,r}/d\vartheta)_b$ is the sum of the horizontal dipole moments per unit azimuthal angle ϑ over all the particles of the b th bunch. The bunch function G takes into account the contribution of the r th particle to the horizontal wake field at the place ϑ_i and at the time t from all its previous turns. The second summation of Eq. (32) has to be made over all the particles of the beam. A similar formula is obtained for the vertical plane by exchanging the roles of the variables x and y in Eq. (32).

In the case of bunched beams with head–tail modes,^{10,11} the equation of betatron motion of the i th particle of the bunch, e.g. in the horizontal plane and in the presence of both wake fields and linear coupling between the transverse planes, is given by¹

$$\frac{d^2x_i}{dt^2} + \left[\omega_{x,i} + \omega_{x0} \left(1 - \frac{\xi_x}{\eta} \right) \frac{d\tau_i}{dt} \right]^2 x_i \\ = \sum_r x_r (t - \tau_r + \tau_i) \chi_x(\tau_r - \tau_i) + \underline{K}_i R^2 \Omega_i^2 y_i. \quad (33)$$

Here, $\xi_x = (dQ_{x,i}/dp_i)(p_0/Q_{x0})$ is the horizontal chromaticity, with p_0 the momentum on the ideal orbit, and $\eta = \gamma_{tr}^{-2} - \gamma^{-2} = -(d\Omega_i/dp_i)(p_0/\Omega_0)$ is the slippage factor. The time-of-arrival of the i th particle at some azimuth measured with respect to the time-of-arrival of the synchronous particle of the bunch is τ_i and $x_r(t - \tau_r + \tau_i) \chi_x(\tau_r - \tau_i)$ is the horizontal wake force on particle i (divided by its mass) due to particle r . A similar

formula is obtained for the vertical plane by exchanging the roles of the variables x and y in Eq. (33).

3.2 Generalised Stability Criteria

It has been shown in Ref. 1 that the two-dimensional dispersion relation is the same for both coasting and bunched beams (with the two particular wake fields considered above), introducing “equivalent” dispersion relation coefficients U_{eq} and V_{eq} ,^{1,8} which are the real betatron frequency shift and instability growth rate respectively, for a given mode, in the absence of coupling and Landau damping. This is perhaps not surprising because in the three cases (Eqs. (1), (32) and (33)) more or less the same differential equation has to be solved.

In the case of coasting beams, U_{eq} and V_{eq} are equal to the coefficients U, V introduced by Laslett *et al.*³ (see Eq. (8)). In the case of bunched beams, U_{eq} and V_{eq} can be deduced from the one-dimensional theory of transverse bunched beam instabilities that has been described in its most general form by Sacherer¹² who combined and extended the results obtained for long and short range interactions.^{9–11,13} The results of coupled Landau damping of transverse bunched beam instabilities (due to coupling impedances $Z_{x,y}$) can be summarised using Sacherer’s general formula for dipole modes, which is as written in Ref. 14

$$\Delta\omega_m^{x,y} = (|m| + 1)^{-1} \frac{j e \beta I_b}{2 m_0 \gamma Q_{x0,y0} \Omega_0 L} \frac{\sum_{k=-\infty}^{k=+\infty} Z_{x,y}(\omega_k^{x,y}) h_m(\omega_k^{x,y} - \omega_{\xi_{x,y}})}{\sum_{k=-\infty}^{k=+\infty} h_m(\omega_k^{x,y} - \omega_{\xi_{x,y}})}, \quad (34)$$

with

$$\omega_k^{x,y} = (k + Q_{x0,y0}) \Omega_0 + m \omega_s. \quad (35)$$

In Eq. (34), $-\infty \leq k \leq +\infty$ for a single bunch or several bunches oscillating independently and $k = n_{x,y} + k' M$ with $-\infty \leq k' \leq +\infty$ for coupled motion of M bunches. Here, $m = \dots, -1, 0, 1, \dots$ is the head–tail mode number, $n_{x,y} = 0, 1, \dots, M-1$ are the coupled-bunch mode numbers related by $n_x = n_y - l$, I_b is the current in one bunch, L is the bunch length (in meters), ω_s is the synchrotron frequency and $h_m(\omega)$ is the

bunch spectrum of mode m . Moreover, $\omega_{\xi_{x,y}} = (\xi_{x,y}/\eta)Q_{x0,y0}\Omega_0$ are the transverse chromatic frequencies. The relations between the “equivalent” dispersion coefficients and the frequency shifts (for mode m), are

$$U_{\text{eq } x,y}^m = \text{Re}(\Delta\omega_m^{x,y}), \quad V_{\text{eq } x,y}^m = -\text{Im}(\Delta\omega_m^{x,y}), \quad (36)$$

where $\text{Im}()$ stands for imaginary part.

With some simplifying assumptions, these results have been verified in Ref. 15 using the Vlasov equation. Therefore, in the case of bunched beams, these coefficients are real constants (for the purpose of the dispersion relation analysis without taking into account space-charge nonlinearities) that characterise both the accelerator and the beam dynamics.

Using U_{eq} and V_{eq} in the case of bunched beams, instead of U and V in the case of coasting beams, the same two-dimensional dispersion relation is obtained (see Eq. (7) or (11)) and the above results remain valid (see Eqs. (20), (21), (26) and (28)).

3.3 Sharing of Both Frequency Spread and Chromaticity

In the case of head–tail instabilities, the transverse betatron frequency shifts, for a given mode m and in the absence of both coupling and Landau damping, are given by Sands for the idealised model in which the wake field of a single particle is zero in front of and constant behind the particle¹⁰ ($\chi_{x,y}(\tau_r - \tau_i) \equiv S_{x,y}$ (positive const.) for $\tau_r > \tau_i$ in Eq. (33) and the similar equation for the vertical plane). It yields

$$U_{\text{eq } x,y}^0 = -\frac{N_b S_{x,y}}{4\omega_{x0,y0}}, \quad U_{\text{eq } x,y}^m = 0 \text{ for } m \neq 0, \quad V_{\text{eq } x,y}^m = -W_{x,y}^m \xi_{x,y}, \quad (37)$$

with

$$W_{x,y}^m = -\frac{2N_b S_{x,y} \hat{\tau}_0}{\pi^2 \eta} (4m^2 - 1)^{-1}. \quad (38)$$

Here, N_b is the number of particles in the bunch and $\hat{\tau}_0$ is the amplitude (in time units) of the synchrotron phase oscillations in the “hollow-bunch model” $\hat{\tau}_i = \hat{\tau}_0$.

To clearly see the effect of coupling on both Landau damping and chromaticity, it is interesting to look at the stability criteria for zero and optimum coupling.¹⁶ In the following, only the most dangerous mode $m=0$ (above transition^b) is considered. In the absence of coupling, the Lorentzian stability criteria in the horizontal and vertical planes respectively, are (see Eqs. (19) and (37))

$$\delta\omega_x + W_x^0 \xi_x \geq 0, \quad \delta\omega_y + W_y^0 \xi_y \geq 0. \quad (39)$$

In the absence of Landau damping, the condition of Sands, $\xi_{x,y} \geq 0$ for the stability of the head–tail mode $m=0$, is recovered (above transition, $\eta > 0$ and thus $W_{x,y}^0 > 0$). With full coupling, the two criteria reduce to (see Eq. (20))

$$\delta\omega_x + \delta\omega_y + W_x^0 \xi_x + W_y^0 \xi_y \geq 0. \quad (40)$$

Equation (40) shows the beneficial effect of coupling as concerns both chromaticity compensation and Landau damping. In the absence of a frequency spread, the result obtained by Talman,¹⁷ $W_x^0 \xi_x + W_y^0 \xi_y \geq 0$, is recovered. A negative chromaticity can be maintained in one plane, provided that the other chromaticity is appropriately increased to compensate. In the presence of Landau damping, it is seen that a less restrictive criterion is obtained. Elliptical spectra exhibit the same kind of results. In the absence of coupling, the transverse stability criteria are (see Eqs. (30))

$$\begin{aligned} \operatorname{Re} \left[\sqrt{\Delta\omega_x^2 - (2U_{\text{eq}x}^0)^2} \right] + 2W_x^0 \xi_x &\geq 0, \\ \operatorname{Re} \left[\sqrt{\Delta\omega_y^2 - (2U_{\text{eq}y}^0)^2} \right] + 2W_y^0 \xi_y &\geq 0. \end{aligned} \quad (41)$$

With optimum coupling, these two criteria reduce to (see Eq. (28))

$$\operatorname{Re} \left[\sqrt{\Delta\omega_x^2 - (2U_{\text{eq}x}^0)^2} + \sqrt{\Delta\omega_y^2 - (2U_{\text{eq}y}^0)^2} \right] + 2(W_x^0 \xi_x + W_y^0 \xi_y) \geq 0. \quad (42)$$

In the absence of a frequency spread, the result of Talman is recovered.

^b The natural chromaticities are (usually) negative and then, to eliminate the most dangerous head–tail mode $m=0$, chromaticity tuning is only necessary above transition energy. The higher head–tail modes, with smaller form factors and wake field coefficients, can be stabilised by Landau damping.

The resistive-wall wake field gives similar results with the “equivalent” dispersion relation coefficients given by Eqs. (26) and (27) of Ref. 11.

3.4 Sharing of Damping by Feedbacks

An electronic feedback system is often used to stabilise a beam in one transverse plane. In the theory of coupled Landau damping, the stabilising effect of a feedback system can be introduced in the general coefficient V_{eq} (see Section 3.2). Its damping effect in one plane can therefore also be transferred to the other plane using coupling.

4 CONCLUSION

The four parameters (horizontal and vertical tune spreads, linear coupling strength and tune distance from the linear coupling resonance $Q_h - Q_v = \text{integer}$) can be used to Landau damp transverse coherent instabilities, with the minimum amount of external non-linearities. If, after applying damping via tune spreads from octupoles, a coherent instability remains in one of the two planes, then linear coupling together with tune separation can transfer Landau damping from the stable to the unstable plane.

In the case of Lorentzian distributions, if the sum of the spreads $\delta\omega_{x,y}$ (half widths at half maximum of the distribution functions) is greater than the sum of the growth rates $V_{\text{eq } x,y}$ (given by Eq. (8) for a coasting beam or Eq. (36) for a bunched beam), then it is possible to stabilise the beam in the two planes by increasing the skew gradient and/or by getting closer to the coupling resonance. The necessary condition $\delta\omega_x + \delta\omega_y \geq V_{\text{eq } x} + V_{\text{eq } y}$ becomes sufficient for full coupling.

However, in practice the coupling has to be optimised because realistic frequency distributions have finite tails. The stability is obtained for coupling values of a certain range. For elliptical spectra, an approximate criterion, valid for optimum coupling, can be expressed as

$$\text{Re} \left(\sqrt{\Delta\omega_x^2 - 4U_{\text{eq } x}^2} + \sqrt{\Delta\omega_y^2 - 4U_{\text{eq } y}^2} \right) \geq 2(V_{\text{eq } x} + V_{\text{eq } y}),$$

where $\text{Re}()$ stands for real part, indicating that the square root has to be omitted if the argument under the root is negative. Here, $\Delta\omega_{x,y}$ are the

half widths at the bottom of the distribution functions and $U_{\text{eq } x,y}$, $V_{\text{eq } x,y}$ are given by Eq. (8) for a coasting beam or Eq. (36) for a bunched beam.

The instability growth rate (which depends on chromaticity for a bunched beam) and damping by feedbacks are “always” transferred between the transverse planes in the presence of coupling. Several parameters could therefore be used to damp transverse coherent instabilities and fix the best working point.

The theory of coupled Landau damping gives a possible explanation of a phenomenon observed in Ref. 18, where a single-bunch instability of the head–tail type has been damped by adjusting $Q_h \approx Q_v$. Furthermore, experiments in the CERN-PS have been performed on a 1 GeV flat bottom, varying the distance from the resonance $Q_h - Q_v = 0$, and the excitation of the skew quadrupoles and octupoles lenses, observing the threshold of a coupled-bunch instability.¹⁹ The results confirm the general behaviour predicted by the theory, considering a Lorentzian distribution in the vertical plane.

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References

- [1] E. Métral, Coupled Landau damping, CERN/PS 97-33 (CA), 1997.
- [2] D. Möhl, On Landau damping of dipole modes by non-linear space charge and octupoles, CERN/PS 95-08 (DI), 1995.
- [3] L.J. Laslett, V.K. Neil and A.M. Sessler, Transverse resistive instabilities of intense coasting beams in particle accelerators, *R.S.I.*, **36**(4), 436–448, 1965.
- [4] B. Zotter, Single beam collective phenomena-transverse (coasting beams), *Proc. First Course Int. School on Accelerators*, Erice, 1976: Theoretical Aspects of the Behaviour of Beams in Accelerators and Storage Rings, CERN 77-13, p. 176, 1977.
- [5] H.G. Hereward, Landau damping by non-linearity, CERN/MPS/DL 69-11, 1969.
- [6] H.G. Hereward, The elementary theory of landau damping, CERN/PS 65-20, 1965.
- [7] A.W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, New York: Wiley, 371 p, 1993.
- [8] E. Métral, Coupled landau damping with elliptical spectra, CERN/PS 98-004 (CA), 1998.
- [9] E.D. Courant and A.M. Sessler, Transverse coherent resistive instabilities of azimuthally bunched beams in particle accelerators, *R.S.I.*, **37**(11), 1579–1588, 1966.
- [10] M. Sands, The head–tail effect: An instability mechanism in storage rings, SLAC-TN-69-8, 1969.
- [11] M. Sands, Head–tail effect II: From a resistive-wall wake, SLAC-TN-69-10, 1969.

- [12] F. Sacherer, Single beam collective phenomena-transverse (bunched beams), *Proc. First Course Int. School on Accelerators*, Erice, 1976: Theoretical Aspects of the Behaviour of Beams in Accelerators and Storage Rings, CERN 77-13, p. 210, 1977.
- [13] C. Pellegrini, *Nuovo Cimento*, **64A**, 477, 1969.
- [14] F. Pedersen, Multibunch instabilities, CERN/PS 93-36 (RF), 1993.
- [15] E. Métral, On coupled Landau damping of transverse bunched beam instabilities, CERN/PS/CA/Note 98-15, 1998.
- [16] E. Métral, Coupled Landau damping of head–tail modes revisited, CERN/PS/CA/Note 98-05, 1998.
- [17] R. Talman, Theory of head–tail chromaticity sharing, CBN-97-16, 1997.
- [18] R. Capii, Observations of high-order head–tail instabilities at the CERN-PS, CERN/PS 95-02 (PA), 1995.
- [19] E. Métral, Measurements of coupled Landau damping of the PS coherent resistive instability, CERN/PS/CA/Note 98-16, 1998.